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A Method to Optimize Weighting of General Planar Arrays

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ABSTRACT

A method is presented which optimizes weights of general planar symmetric arrays. It applies to full regular arrays as well as sparse arrays with perturbed positions.

The objective is to find a weighting of the array elements which gives the minimum sidelobe level of the array pattern in a specified region - the stopband. The sidelobe level is controlled on a discrete set of points from this region. The method minimizes the Chebyshev norm of the sidelobe level.

The method is based on linear programming and is solved with the standard simplex method. Examples of optimal weighted 1D and 2D planar arrays are presented.

1. INTRODUCTION

In array signal processing we are interested in signals conveyed by propagating waves. To extract these signals, we apply an array which is a group of sensors located at distinct spatial locations. With these sensors we can measure the wavefield at the sensor locations. By processing these measurements we may extract only those signals we are interested in. This is done by filtering both in time and space.

In this article we will concentrate on general symmetric planar arrays. By general, we mean full regular arrays as well as sparse arrays with perturbed positions. The latter arises when we remove some array elements, and break the regular $\lambda/2$ interelement spacing. This is motivated from the potential for improved resolution or reduced sidelobe levels compared to an equispaced array with the same number of elements [2].

1.1. Array pattern

From classical signal processing, a linear shift-invariant system is often characterized by its frequency response $H(e^{j\omega T})$,

$$H(e^{j\omega T}) = \sum_{n=1}^{2N} h_n e^{-jn\omega T} \tag{1}$$

In array signal processing the array pattern $W(\vec{k})$ plays much the same role in characterizing an array's performance [3]. It corresponds to the wavenumber frequency response of the spatiotemporal filter. The array pattern of a $2N$ element array is given as

$$W(\vec{k}) = \sum_{n=1}^{2N} w_n e^{j\vec{k}\cdot\vec{x}_n} \tag{2}$$

where the array element locations are $\vec{x}_n \in \mathbb{R}^3$ with the corresponding weights $w_n \in \mathbb{R}$. The wavenumber vector $\vec{k} \in \mathbb{R}^3$ has amplitude $|\vec{k}| = 2\pi/\lambda$ where λ is the design wavelength.

Consider a regular linear array with element spacing $\Delta x = \lambda/2$. The vector-product in the array pattern (2) then simplifies to

$$\vec{k}\cdot\vec{x}_n = |\vec{k}||\vec{x}_n|\cos\alpha = \frac{2\pi}{\lambda}\, n\Delta x\,(-u)$$

where $u = \sin\phi = -\cos\alpha$ for ϕ given in figure 1. In this special case the array pattern is

$$W(u) = \sum_{n=1}^{2N} w_n e^{-jn\pi u} \tag{3}$$

The relationship between the array pattern and a filter's frequency-response is now obvious as we have the following correspondance

$$\begin{array}{ccc} \omega T & \leftrightarrow & \pi u \\ h_n & \leftrightarrow & w_n \end{array}$$

1.2. Symmetry

To ensure a real array pattern and a real optimization problem, we consider symmetric arrays with

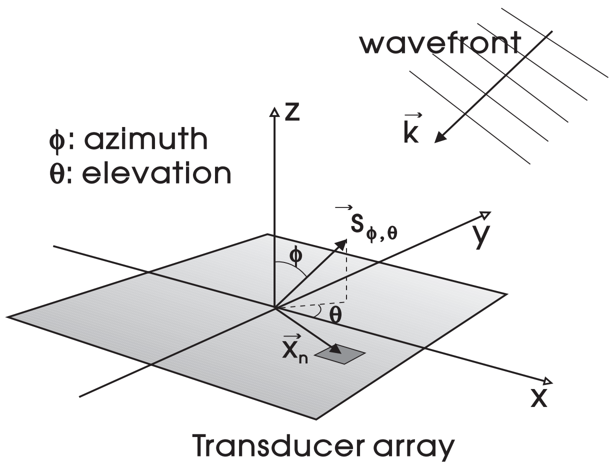

Figure 1. A 2D planar array with spherical and rectangular coordinates.

symmetric real weighting. The following equalities must then be satisfied for $w_n \in \mathbb{R}$

$$\left.\begin{array}{rcl} w_{N+n} & = & w_n \\ \vec{x}_{N+n} & = & -\vec{x}_n \end{array}\right\} \quad n = 1, \cdots, N \tag{4}$$

By combining (2) and (4) we can write the array pattern as a sum of cosines

$$W(\vec{k}) = 2\sum_{n=1}^{N} w_n \cos(\vec{k}\cdot\vec{x}_n) \tag{5}$$

It is convenient to introduce a unit direction vector in spherical coordinates $\vec{s}_{\phi,\theta} \in \mathbb{R}^3$ as

$$\vec{s}_{\phi,\theta} = -\frac{\vec{k}}{|\vec{k}|} \tag{6}$$

Substituting this for $\vec{k}$ in (5) and using the fact that $\cos\alpha = \cos(-\alpha)$, we get the directive array pattern as

$$W(\phi,\theta) = 2\sum_{n=1}^{N} w_n \cos\left(\frac{2\pi}{\lambda}\vec{s}_{\phi,\theta}\cdot\vec{x}_n\right) \tag{7}$$

which gives the array response to a monochromatic wave from direction (ϕ,θ) in space. See figure 1.

1.3. A planar array

The elements of a planar array are located in the xy-plane. There are both 1D and 2D planar arrays. The 1D linear array is a special case of the 2D planar array, since the array elements are restricted to locations on the x-axis.

For a 2D planar array the rectangular coordinates of array element n is $\vec{x}_n = (x_n, y_n, 0)$. The unit direction vector from (6) has the rectangular coordinates $\vec{s}_{\phi,\theta} = (\sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi)$. The vector-product in (7) then becomes

$$\vec{s}_{\phi,\theta}\cdot\vec{x}_n = \sin\phi(x_n\cos\theta + y_n\sin\theta) \tag{8}$$

and the array pattern for the general 2D planar array is

$$W(\phi,\theta) = 2\sum_{n=1}^{N} w_n \cos\left(\frac{2\pi}{\lambda}\sin\phi(x_n\cos\theta + y_n\sin\theta)\right) \tag{9}$$

where $(x_n, y_n, 0)$ is the rectangular coordinate of array element n with the corresponding weight $w_n \in \mathbb{R}$. λ is the design wavelength and (ϕ,θ) gives the array's spherical look direction.

It is convenient to write the array pattern (9) in matrix notation. We can write it as

$$W(\phi,\theta) = \mathbf{v}(\phi,\theta)^T\mathbf{w} \tag{10}$$

where $\mathbf{w} = [w_1 \ \cdots \ w_N]^T$ are the element weights and the kernel vector $\mathbf{v}(\phi,\theta)$ is given as $\mathbf{v}(\phi,\theta) = \left[2\cos(\frac{2\pi}{\lambda}\vec{s}_{\phi,\theta}\cdot\vec{x}_1) \ \cdots \ 2\cos(\frac{2\pi}{\lambda}\vec{s}_{\phi,\theta}\cdot\vec{x}_N)\right]^T$ with the vector product $\vec{s}_{\phi,\theta}\cdot\vec{x}_n$ as defined in (8).

2. OPTIMIZATION PROBLEM

When such arrays are utilized in beamforming, it is desired that the array pattern consists of a narrow mainlobe and a low sidelobe level when the array is looking along the z-axis. When the array elements are fixed, we can optimize the array weighting due to this criterion.

The idea is to minimize the sidelobe level in a continious region $\mathcal{R}$ of the $\phi\theta$-plane. See figure 2. We want to suppress signals from these directions, thus this region corresponds to a filter's stopband. Remark that the array pattern is symmetric about the θ-axis; $W(-\phi,\theta) = W(\phi,\theta)$. This leaves us with only the right halfplane in optimization. The passband in this case is minimal in the sense that it consists only of the θ-axis.

2.1. Optimization formulation

The optimization problem may now be stated loosely as

$$\begin{array}{ll} \underset{\mathbf{w}}{\textit{Minimize}} & \text{array pattern level in stopband} \\ \\ \textit{Subject to} & \text{normalized array pattern level} \\ & \text{in passband} \end{array} \tag{11}$$

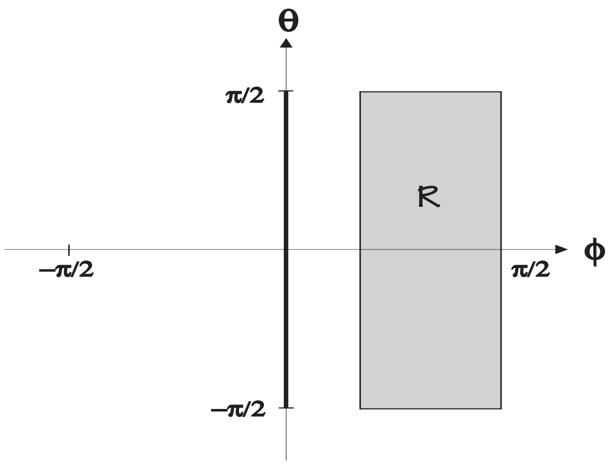

Figure 2. The optimization region $\mathcal{R}$ in the $\phi\theta$-plane.

The constraint of a normalized array pattern level in the passband is written as

$$W(0,\theta) = 2\sum_{n=1}^{N} w_n = 1 \tag{12}$$

The stopband region $\mathcal{R}$ is then discretisized into M gridpoints $R = \{(\phi_1,\theta_1)\ \cdots\ (\phi_M,\theta_M)\}$. The absolute array pattern level δ_s on the discrete set R is defined as

$$\delta_s = \max_{\phi,\theta\in R} |W(\phi,\theta)| \tag{13}$$

A more formal optimization formulation of (11) is now obtained with (12) and (13)

$$\begin{array}{ll} \underset{\mathbf{w}}{\mathit{Minimize}}\ \delta_s & \\ \\ \mathit{Subject\ to} & W(0,\theta) = 1 \\ & |W(\phi,\theta)| \le \delta_s \quad \forall\ (\phi,\theta) \in R \end{array} \tag{14}$$

This optimization problem can be transformed into one of the standard linear programming matrix forms

$$\begin{array}{ll} \mathit{Minimize} & \mathbf{c}^T\mathbf{x} \\ \\ \mathit{Subject\ to} & \mathbf{A}\mathbf{x} \le \mathbf{b} \end{array} \tag{15}$$

The problem in (14) may be written in standard form (15) by introducing block matrices for $\mathbf{c}, \mathbf{x}, \mathbf{A}$ and $\mathbf{b}$. Let the variable vector $\mathbf{x}$ consist of the weights $\mathbf{w}$ and the array pattern level indicator δ_s. The full linear program is stated as

$$\underset{\mathbf{w}}{\mathit{Minimize}} \qquad \begin{bmatrix} \mathbf{0}_N^T & 1 \end{bmatrix} \begin{bmatrix} \mathbf{w} \\ \delta_s \end{bmatrix}$$

$$\mathit{Subject\ to}$$

$$\begin{bmatrix} \mathbf{1}_N^T & 0 \\ \mathbf{-1}_N^T & 0 \\ \mathbf{v}(\phi,\theta)^T & -1 \\ -\mathbf{v}(\phi,\theta)^T & -1 \end{bmatrix} \begin{bmatrix} \mathbf{w} \\ \delta_s \end{bmatrix} \le \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} \tag{16}$$

$$\forall\ (\phi,\theta) \in R$$

2.2. Duality

Every linear program has another program associated with it. One of them is called the *primal* problem and the other is called the *dual* problem. From linear programming theory, the duality theorem assures us that if an optimal solution exists to either of them, then the other also has an optimal solution and the objective value coincides.

Since the solution to both programs are obtained by solving either one, it may be advantageous to solve the dual program rather than the primal itself.

The full linear program in (16) has an $\mathbf{A}$ matrix with $2M+2$ rows and $N+1$ columns. M is the number of discrete points on R and N are half the $2N$ element weights by symmetry. For most purposes $M > N$. With this kind of problem it is more effective to solve its dual [1]

$$\begin{array}{llll} \mathit{Maximize} & \mathbf{b}^T\mathbf{y} & & \\ \\ \mathit{Subject\ to} & \mathbf{A}^T\mathbf{y} & = & \mathbf{c} \\ & \mathbf{y} & \le & \mathbf{0} \end{array} \tag{17}$$

where $\mathbf{c}, \mathbf{x}, \mathbf{A}$ and $\mathbf{b}$ is as above. The optimal solution to the primal is established as a transform of the optimal dual solution. Let $\mathbf{y}^*$ be the optimal solution to the dual problem. Then the optimal solution $\mathbf{x}^*$ to the primal problem is

$$\mathbf{x}^* = \mathbf{A}_0^{-1}\mathbf{b} \tag{18}$$

In (18) $\mathbf{A}_0$ is the rows from $\mathbf{A}$ corresponding to the *basic* variables in $\mathbf{y}^*$. These are the nonzero elements in $\mathbf{y}^*$.

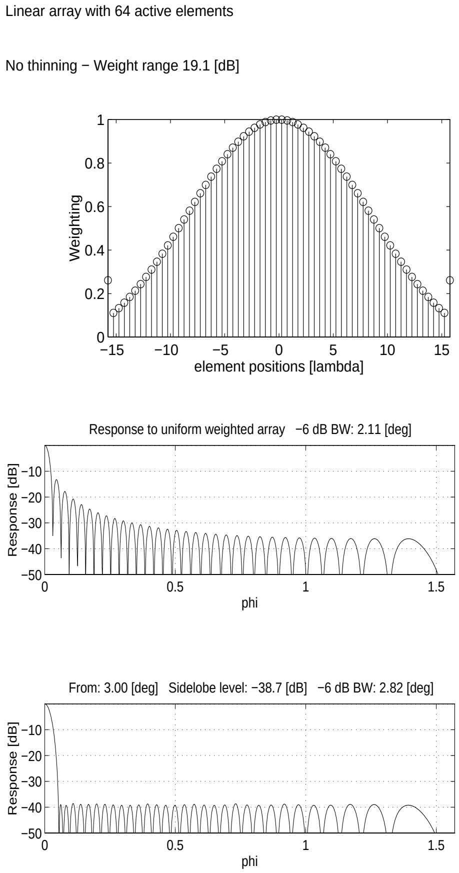

Figure 3. The optimal weights and the array response to a 1D regular array. The response to the unweighted array is given as a reference.

3. EXAMPLES

The following examples were all set up with MATLAB and solved using the general linear optimizer CPLEX. The problems were solved on a DEC-5000/200.

3.1. 1D regular array

First a linear array with 64 elements was optimized. The stopband region was $\phi \in [3, 90]$ degrees. The optimal weights and array response is plotted in figure 3. The characteristic Chebyshev weighting spikes appear at each end element. With $M = 512$ gridpoints, the optimal weights were obtained after 7.0 seconds.

3.2. 1D sparse and perturbed array

Then a 128 element array was thinned 25 % and the elementpositions perturbed. The stopband region was $\phi \in [2, 90]$ degrees. The optimal weights and array response is plotted in figure 4. This optimization took 13.4 seconds with $M = 512$ gridpoints.

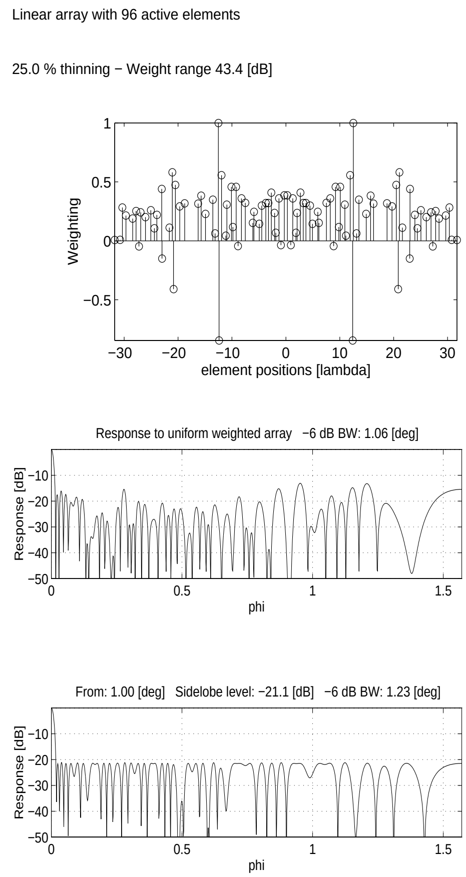

Figure 4. The optimal weights and the array response to both a uniform and an unweighted 1D irregular array.

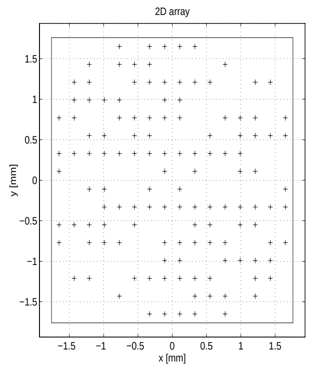

Figure 5. Element distribution of the 2D sparse array.

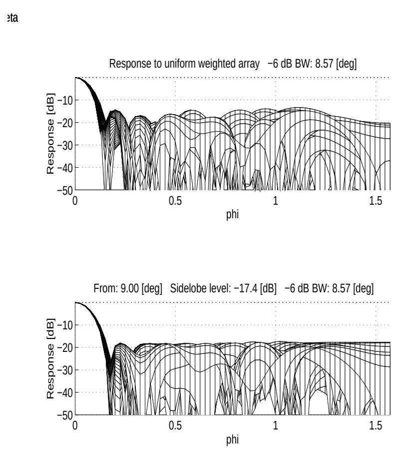

Figure 6. The array response to both a uniform and an unweighted 2D sparse array, viewed from side.

3.3. 2D sparse array

Finally a 256 element array was thinned 50 %. The element positions are given in figure 5. The stop-band region was $\phi \in [9, 90]$ and $\theta \in [-\pi/2, \pi/2]$ degrees. The optimal array response is plotted in figure 6, and was obtained after 30.8 seconds with $M = 1024$ gridpoints.

4. CONCLUSION

The method was shown to converge for three very different arrays within resonable time. The formulation is simple, and it would be a an easy task for instance to restrict it to positive weights.

The method will also apply to both 1D and 2D filter design with slight modifications.

The problem formulation is in a standard linear programming form and is solved effectively by the Simplex method. This is advantageous since there are many implementations available.

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